# 2-envelopes and the analytic hierarchy

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The main purpuse with this note is to prove that $\Pi^1_n$ over the reals will always be the 2-envelope of some functional, while $\Sigma^1_n$ will never be. Through the investigation we were led to a lemma on inductive definitions, which seem to be of general interest. We were also led to a study of the weak type 3 functional $W$ defined by $\forall F(W(F) = 0)$.

## 1. Relativized $\Pi^1_1$-inductive definitions over the reals.

Let $\omega$ = the natural numbers, $R = {}^{\omega}\omega$, and $\mathcal{R}$ the powerset of $R$. We say that a predicate on $\omega^n \times R^m \times \mathcal{R}^k$ is simple if it is defined by :

i Using the following predicates: $\in$ in $R \times \mathcal{R}$ $(\alpha \in A)$, Evaluation in $R$, $\alpha(n) = m$, and in addition : All primitive recursive operations on $R$ and $\omega$ may be used as functions.

ii Using the connectives $\vee$ and $\neg$.

We use $A, B, C, S$ for elements in $\mathcal{R}$, $\alpha,\beta$ for elements in $R$, $i, j, k, m, n$ for natural numbers. $\gamma,\delta$ and $\lambda$ will denote ordinals. We do not distinguish between the use of a letter as a variable and as a name on a given object.

Let $\varphi$ be a simple predicate. We say that a variable $A$ occurs positively in $\varphi$ if all subformulas $t \in A$ occur positively in $\varphi$, where $t$ is a term.

We say that a set $\Omega$ in $\omega^n \times R^m \times \mathcal{R}^k$ is $\Pi^1_1$ if there is a simple predicate $\varphi$ such that $X \in \Omega \iff \forall\alpha\, \exists n\, \varphi(\alpha,n,X)$.

By standard methods we prove that the $\Pi^1_1$-sets are closed under disjunction, conjunction, $\forall^\omega$, $\exists^\omega$ and $\forall^R$. Moreover, if $\psi$ is arithmetic with simple matrix, then there is a $\Pi^1_1$-formula $\psi'$ equivalent to $\forall\alpha\psi$.

Remark: The simple sets will be primitive Kleene-recursive subsets of $\omega^n \times R^m \times \mathcal{R}^k$.

A set $A \subseteq R$ is $\Pi^1_1(S)$ if there is a $\Pi^1_1$-set $\Omega$ in $R \times \mathcal{R}$ such that

$$\forall\alpha(\alpha \in A \iff \langle\alpha,S\rangle \in \Omega)$$

$A$ is positive in $S$ if $S$ occurs positive in the definition of $\Omega$.

Definition. Let $\Gamma\colon \mathcal{R} \to \mathcal{R}$ and $S \in \mathcal{R}$. We say that $\Gamma$ is a $\Pi^1_1(S)$-operator if there is a simple predicate $\varphi$ such that

$$\alpha \in \Gamma(A) \iff \forall\beta\,\exists n\,\varphi(n,\alpha,\beta,A,S)$$

$\Gamma$ is positive if $A$ occurs positively in $\varphi$. If also $S$ occurs positively, we say that $\Gamma$ is positive in $S$.

We use some standard notation for inductive definitions :

$$\Gamma^0 = \emptyset\,, \quad \Gamma^{\gamma+1} = \Gamma^\gamma \cup \Gamma(\Gamma^\gamma)$$

$$\Gamma^\gamma = \bigcup_{\gamma<\lambda} \Gamma^\gamma \text{ when } \lambda \text{ is a limit.}$$

$$\Gamma^\infty = \bigcup_{\Gamma\in On} \Gamma^\gamma$$

Remark: When $\Gamma$ is positive, $\Gamma$ will be monotone and thus $\Gamma$ will have a least fixed point $A = \Gamma^\infty$ such that $\Gamma(A) \subseteq A$.

$\Pi^1_1(S)$ will not in general be normed. When it is, we can use a theorem of Moschovakis [3] and prove that if $\Gamma$ is a positive $\Pi^1_1(S)$ operator, then $\Gamma^\infty$ will be $\Pi^1_1(S)$. We will prove that this is the case for arbitrary $S$.

<u>Definition.</u> Let $\Gamma$ be a $\Pi^1_1(S)$ operator defined as above. Let $B \subseteq R$. We define the <u>operator $\Gamma_B$</u> by

$$\alpha \in \Gamma_B(A) \Longleftrightarrow \forall\beta \in B\ \exists n\ \varphi(n,\alpha,\beta,A,S) \quad \& \quad \alpha \in B$$

<u>Remark:</u> If $\Gamma$ is positive, then $\Gamma_B$ will be monotone. For arbitrary $\Pi^1_1(S)$ operations $\Gamma$:

$$\text{If } B_1 \supseteq B_2, \text{ then } \Gamma_{B_1}(A) \cap B_2 \subseteq \Gamma_{B_2}(A) \subseteq B_2$$

$$\text{For all } B \subseteq R, \quad \Gamma(A) \cap B \subseteq \Gamma_B(A)$$

<u>Definition.</u> Let $\alpha_1,\cdots,\alpha_n \in R$. Then <u>$[\alpha_1,\cdots,\alpha_n]$</u> consists of all reals primitive recursive in the sequence $\langle\alpha_1,\cdots,\alpha_n\rangle$.

<u>Remark:</u> Let $\varphi,\alpha$ and $\beta$ be given, $\varphi$ simple. Then for all $A,S$

$$\exists n\varphi(n,\alpha,\beta,A,S) \Longleftrightarrow \exists n\varphi(n,\alpha,\beta,A \cap [\alpha,\beta],\ S \cap [\alpha,\beta])$$

since we in the verification of $\varphi(n,\alpha,\beta,A,S)$ only ask questions about $A$ and $S$ which are primitive recursive in $\alpha$ and $\beta$.

We now have sufficient observations to prove the following lemma:

<u>Lemma 1.</u> Let $\Gamma$ be a positive $\Pi^1_1(S)$-operator, $S \in \mathcal{R}$. Let $\gamma \in On$. Let $B \subseteq R$ be countable and closed under primitive recursion. Let $\alpha \in B$. Then $\alpha \in \Gamma^\gamma \Rightarrow \alpha \in \Gamma^\gamma_B$.

Proof. Let B be countable and closed under primitive recursion.

We prove the lemma by induction on $\gamma$, i.e. $\Gamma^{\gamma} \cap B \subseteq \Gamma_B^{\gamma}$, $\Gamma^0 = \Gamma_B^0 = \emptyset$. At limit stages the induction is trivial.

Let $\alpha \in B$ and assume

$$\forall\beta\exists n\varphi(n,\alpha,\beta,\Gamma^{\gamma},S)$$

Now, let $\beta \in B$ and choose $n$ such that

$$\varphi(n,\alpha,\beta,\Gamma^{\gamma},S)$$

Since $[\alpha,\beta] \subseteq B$, we have

$$\varphi(n,\alpha,\beta,\Gamma^{\gamma} \cap B,S)$$

By induction hypothesis, $\Gamma^{\gamma} \cap B \subseteq \Gamma_B^{\gamma}$, and since the operator is positive, we have

$$\varphi(n,\alpha,\beta,\Gamma_B^{\gamma},S)$$

But $\beta \in B$ was arbitrary, so $\alpha \in \Gamma_B(\Gamma_B^{\gamma}) \subseteq \Gamma_B^{\gamma+1}$.

We of course obtain that $\Gamma^{\infty} \cap B \subseteq \Gamma_B^{\infty}$ for all countable B closed under primitive recursion.

□

We are able to reverse lemma 1.

<u>Lemma 2.</u> Let $\Gamma$ be a positive $\Pi^1_1(S)$-operator, $S \in \mathcal{R}$. Let $\alpha \in R$ and assume $\alpha \notin \Gamma^\infty$.

Then there is a countable subset $B$ of $R$ such that $\alpha \in B$, $B$ is closed under primitive recursion and $\alpha \in \Gamma^\infty_B$ .

<u>Proof.</u> The idea is as follows: Assume $\alpha \notin \Gamma^\infty$. By Skolem-Løwenheim's theorem there is a countable transitive structure $M$ such that $M \models \alpha \notin \Gamma^\infty$. Let $B = M \cap R$. Then $(\Gamma^\infty)_M = \Gamma^\infty_B$ , and $\alpha \notin \Gamma^\infty$ .

We will now give a proof along these lines. Let

$$\mathcal{V} = \langle V_K, \in, S, \alpha \rangle \;, \quad \text{where} \quad K > |\Gamma|$$

Let $\langle A_\delta \rangle_{\delta \leq \gamma}$ be a sequence. By $\langle A_\delta \rangle_{\delta \leq \gamma}$ is $\Gamma$-inductive we mean the following statement :

$$\forall \delta < \gamma \forall \beta [(\beta \in A_{\delta+1} \Leftrightarrow \forall \beta' \exists n \varphi(n, \beta, \beta', A_\delta, S) \vee \beta \in A_\delta)]$$

$$\wedge \; \forall \lambda \leq \gamma (\lambda \text{ limit} \Rightarrow A_\lambda = \bigcup_{\delta < \lambda} A_\delta)$$

$$\wedge \; A_0 = \emptyset \; .$$

The following two formulas will be valid in $\mathcal{V}$:

I $\quad \forall \gamma \in On \;\; \forall \langle A_\delta \rangle_{\delta \leq \gamma} \;\; (\langle A_\delta \rangle_{\delta \leq \gamma}$ is $\Gamma$-inductive $\Rightarrow \alpha \notin A_\gamma)$

II $\quad \exists \gamma \in On \;\; \exists \langle A_\delta \rangle_{\delta \leq \gamma+1} \;\; (\langle A_\delta \rangle_{\delta \leq \gamma+1}$ is $\Gamma$-inductive $\wedge \; A_{\gamma+1} = A_\gamma)$.

By Skolem-Löwenheim's theorem, let $M'$ be a countable substructure of $\mathcal{V}$, elementary equivalent to $\mathcal{V}$. Let $M$ be the transitive structure obtained from $M'$ by Mostowskis isomorphism theorem. Since all natural numbers are in $M'$, $M \cap R = M' \cap R$. $M$ will be elementary equivalent to $\mathcal{V}$, and thus formulas I and II hold in M. Let $B = M \cap R$. B is countable and closed under primitive recursion.

Now, let $\gamma$ and $\langle A_\delta \rangle_{\delta \leq \gamma+1}$ come from II in M. $(\langle A_\delta \rangle_{\delta \leq \gamma+1}$ is $\Gamma$-inductive$)_M$ means :

$$\forall \delta < \gamma+1 \quad \forall \beta \in B(\beta \in A_{\delta+1} \Longleftrightarrow \forall \beta' \in B\ \exists n \varphi(n,\beta,\beta',A_\delta,(S)_M)$$

$$\wedge \lambda\ \lambda \leq \gamma+1 \quad (\lambda \text{ limit} \Rightarrow A_\lambda = \bigcup_{\delta < \partial} A_\delta)$$

$$\wedge A_0 = \emptyset .$$

Here $(S)_M = S \cap B$. Thus $A_\delta = \Gamma_B^\delta$ for all $\delta \in On \cap M$

From I we have that $\alpha \notin A_{\gamma+1} = {\Gamma_B}^{\gamma+1} = {\Gamma_B}^\infty$

This ends the proof.

Remark: If $\gamma$ is a countable ordinal we have

$$\alpha \in \Gamma^\gamma \Longleftrightarrow \forall B\ (B \text{ is countable } \wedge \alpha \in B \wedge B \text{ is closed under primitive recursion } \Rightarrow \alpha \in {\Gamma_B}^\gamma)$$

This is either proved by induction on $\gamma$ or by a proof similar to that of lemma 2.

If $\gamma$ is not countable, the statement is false if and only if $|\Gamma| > \gamma$.

Theorem 1. Let $\Gamma$ be a positive $\Pi_1^1(S)$-operator, $S \in \mathcal{R}$. Then $\Gamma^\infty$ is $\Pi_1^1(S)$. Moreover, if $\Gamma$ is positive in S, then $\Gamma^\infty$ is positive in S.

<u>Proof.</u> By lemma 1 and 2 we have

$$\alpha \in \Gamma^{\infty} \iff \forall B(\alpha \in B \ \& \ B \text{ is countable } \& \ B \text{ is closed under primitive recursion} \Rightarrow a \in \Gamma_B^{\infty}).$$

All $\Gamma_B$'s will be monotone operators.

Let $\beta \in R$, $B_\beta = \{\beta_n;\ n \in \omega\}$, where we use some standard primitive recursive decomposition of a real to a sequence of reals. The predicate ' $\alpha \in B_\alpha$ & $B_\alpha$ is closed under primitive recursion ' will be arithmetic. We claim that $\Gamma_{B_\beta}^{\infty}$ will be $\Pi_1^1(S,\beta)$ uniform in $\beta$.

We translate $\Gamma_{B_\beta}$ to the inductive definition $\Gamma_0$ on $\omega$ by

$$n \in \Gamma_0(T) \iff \beta_n \in \Gamma_{B_\beta}(\{\beta_m; m \in T\})$$

Then

$$n \in \Gamma_0^{\infty} \iff \forall T(\forall m(m \in \Gamma_0(T) \Rightarrow m \in T) \Rightarrow n \in T)$$

Since the expression '$m \in \Gamma_0(T)$' occurs positively here, this will be $\Pi_1^1(S,\beta)$, uniform in $\beta$, and positive in $S$ if $\Gamma$ is positive in $S$.

$\Gamma_0^{\infty}$ may be translated back to

$$\alpha \in \Gamma_{B_\beta}^{\infty} \iff \exists n(\alpha = \beta_n \ \& \ n \in \Gamma_0^{\infty}).$$

This ends the proof of the claim.

Now

$$\alpha \in \Gamma^{\infty} \iff \forall \beta(B_\beta \text{ is closed under primitive recursion } \& \ \alpha \in B_\beta \Rightarrow \alpha \in \Gamma_{B_\beta}^{\infty})$$

This is $\Pi_1^1(S)$ and positive in $S$ if $\Gamma$ is.

## 2. Recursion in W.

Definition. Let $W$ be the type-3-functional defined by $\forall F(W(F) = 0)$.

Theorem 2. Let $S$ be a functional, the type of $S \leq 2$. Then

$$2\text{-sc}(W,S) = 2\text{-sc}(S)$$

$2\text{-en}(W,S) = \forall^R(2\text{-en } S)$ = The least family including 2-en(S) that is closed under universal quantifiers over the continuum.

Proof. Let $S$ be as in the theorem. First we prove that $2\text{-sc}(S) = 2\text{-sc}(S,W)$. The idea is that $W$ only checks totality, and thus total $S,W$-computable functions will be $S$-computable. To be more precise, we find a primitive recursive function $f$ such that when $\{e\}^{S,W}(x) \simeq m$, then $\{f(e)\}^S(x) \simeq m$. Then, if $\{e\}^{S,W}$ is a total function, $\{f(e)\}^S$ will be the same total function.

Define $g(e,s)$ in the following way :

When $e$ codes initial computations, let $g(e,s) = e$.
When $e$ is an index for composition, i.e. $e = \langle 4,n,e_1,e_2\rangle$, let $g(e,s) = \langle 4,n,g(e_1,s),g(e_2,s)\rangle$.

The cases when $e$ is an index for primitive recursion or permutation are treated similarly.

If $e$ is an index for the schema for application of $W$ then $\{e\}^{S,W}(e',a) \simeq 0 \iff \lambda\beta\{e'\}^{S,W}(\beta,a)$ is total. Let $g(e,s)$ be an $S$-index for the function $h$ where $h(e',a) = 0$ for all $e',a$.

If $e = \langle 9,n,m\rangle$ (schema for diagonalization), let $g(e,s) = t(s)$ where $t$ is a primitive recursive function such that for all $s,x,a$ :

$$\{\{s\}_{pr}(x)\}^S(a) \simeq \{t(s)\}^S(x,a,b)$$

By the recursion theorem for primitive recursion, there is a number $s$ such that for all $e$ :

$$g(e,s) = \{s\}_{pr}(e)$$

Let $f(e) = g(e,s)$ for this $s$. By induction on the length of the computation $\{e\}^{S,W}(x) \simeq m$ we prove that $\{f(e)\}^S(x) \simeq m$.

Interesting cases :

1. $e$ is an index for the schema for application of $W$. Suppose $\{e\}^{S,W}(e',a) \simeq 0$.
   $\{f(e)\}^S(e',a) \simeq 0$ because $\{f(e)\}^S(e',a) \simeq h(e',a) = 0$
2. $e = \langle 9,n,m\rangle$. Suppose $\{e\}^{S,W}(e',a,b) \simeq m$. Then $\{e'\}^{S,W}(a) \simeq m$. By induction hypothesis : $\{f(e)\}^S(a) \simeq m$. Since $f(e') = g(e',s) = \{s\}_{pr}(e')$ : $\{\{s\}_{pr}(e')\}^S(a) \simeq m$. Hence $\{t(s)\}^S(x,a,b) \simeq m$. Since $f(e) = g(e,s) = t(s)$, $\{f(e)\}^S(x,a,b) \simeq m$.

This proves the induction, and hence that $2\text{-sc}(S) = 2\text{-sc}(S,W)$.

<u>Remark:</u> This is a reindexing. See Harrington [1] for definition.

Then we prove that 2-en(S,W) is closed under $\forall^R$. Let $A \subseteq R \times R$ be S,W semi computable. Let $\alpha \in B \Longleftrightarrow \forall\beta\ \langle\alpha,\beta\rangle \in A$.

Let $\varphi$ be S,W-computable such that

$$\varphi(\langle\alpha,\beta\rangle) \downarrow \Longleftrightarrow \langle\alpha,\beta\rangle \in A$$

Define the S,W-computable function $\psi$ by

$$\psi(\alpha) = W(\lambda\beta\varphi(\alpha,\beta))$$

Then

$$\psi(\alpha) \downarrow \Longleftrightarrow \lambda\beta\varphi(\alpha,\beta) \text{ is total } \Longleftrightarrow \forall\beta\varphi(\alpha,\beta) \downarrow \Longleftrightarrow \alpha \in B$$

What is left to prove is that the set of W,S-computations is an $\forall^R$(2-en S)-set. We will, in fact prove that the set of W,S-computations is $\Pi^1_1(S)$.

A proper investigation of Kleene's schemata [2] gives that the W,S-computations are given by a positive $\Pi^1_1(S)$-operator. We may then use theorem 1.

The definition of the operator is trivial except in the following cases :

<u>Composition.</u> $\Phi(b) = \psi(X(b),b)$

Given indices $e_1, e_2$ and $e_3$ for $\Phi, \psi$ and $X$ we have

$$\langle e_1,b,n\rangle \in \Gamma(A) \Longleftrightarrow \exists m(\langle e_2,m,b,n\rangle \in A \ \&\ \langle e_3,b,m\rangle \in A)$$

<u>Case 8.</u> $\Phi(\alpha^j,b) = \alpha^j(\lambda\alpha^{j-2}X(\alpha^j,\alpha^{j-2},b))$

If $j = 3$, there is only $W$ to use. Let $\hat{W}$ be a fixed index for W, and $e$ the index for $X$. Then

$$\langle\hat{W},e,b,n_0\rangle \in A \Longleftrightarrow \forall n\ \exists n\ \langle e,n,b,m\rangle \in A$$

$$\&\ \forall\beta(\forall n,m(\langle e,n,b,m\rangle \in A \Rightarrow \beta(n) = m) \Rightarrow n_0 S(\beta)).$$

Here A occurs positively, and the claim and theorem is established.

□

Corollary 1. Let $n \in \omega$.

$\Pi^1_n$ on $R$ is the 2-envelope of some type $\leq 3$ functional.

Proof. If $n = 1$, this is known $\Pi^1_1 = 2\text{-en}({}^2E)$. If $n > 1$, let $S$ be the characteristic function of a complete $\Sigma^1_{n-1}$-set of reals. Obviously $\Sigma^1_{n-1} \leq 2\text{-en}(S)$. Given $\alpha \in R$, the set of Kleene-computations over $\omega$ in $S$ and $\alpha$ is given by a $\Delta^1_1(S,\alpha)$ inductive definition, uniform in $\alpha$. In fact, all cases will be arithmetic, except case 8, recursion in $S$. Here the $\Pi$-form is given in the proof of theorem 2, the $\Sigma$-form will be

$$\langle\hat{S},e,b,n_0\rangle \in A \Longleftrightarrow \forall n\, \exists m\, \langle e,n,b,m\rangle \in A$$

$$\& \; \exists\beta(\forall m\, \langle e,m,b,\beta(m)\rangle \in A \;\&\; S(\beta) = n_0).$$

Since WO (well-ordering of $\omega$) is arithmetic in $S$, the set of Kleene-computations over $\omega$ in $S,\alpha$ will be $\Delta^{1\,\alpha}_n$, uniform in $\alpha$. So $2\text{-en}(S) \subseteq \Delta^1_n$.

$$\forall^R(\Sigma^1_{n-1}) = \forall^R(\Delta^1_n) = \Pi^1_n \text{ , so}$$

$$2\text{-en}(S,W) = \forall^R(2\text{-en } S) = \Pi^1_n \text{ .}$$

Corollary 2. $\Sigma^1_n$ is not the 2-envelope of any functional.

Proof. For functionals of type $\leq 2$, this was proved by Moschovakis in [3], and independently by Kechris. Let $f_p(U) \geq 3$. Then $W$ is recursive in $U$. From the proof of theorem 2 we see that 2-en(U) is closed under $\forall^R$. This is not the case for $\Sigma^1_n$, so $2\text{-en}(U) \neq \Sigma^1_n$.

Corollary 3.

$2\text{-en}(W) = 2\text{-en}(W,{}^2E) = 2\text{-en}({}^2E) = \Pi^1_1$

$2\text{-sc}(W) =$ the recursive sets

$2\text{-sc}(W,{}^2E) = 2\text{-sc}({}^2E) = \Delta^1_1$ .

For any functional $U$, let $Th(U)$ denote the Kleene theory of $U$ over $R$ with associated length function. We see that $Th({}^2E)$ and $Th({}^2E,W)$ have the same 2-envelopes and the same 2-section. In both theories we have arbitrary long countable computations. However, if $S$ is arbitrary tp $\leq 2$ we shall see that in $Th(S,W)$ it is a 'quick' operation to check that a tuple is a computation.

The set of computations in $Th(S,W)$ is given by a $\Pi^1_1(S)$-expression.

$$\sigma \text{ is a computation} \iff \forall\alpha\ \exists n\ \varphi(\alpha,n,\sigma,S)$$

where $\varphi$ is simple. Given $\alpha,n$ and $\sigma$ we may effectively in $\alpha,\sigma,S$ decide whether $\varphi(\alpha,n,\sigma,S)$ holds or not. Thus there is a S-recursive function f such that $f(\alpha,\sigma)\downarrow \iff \exists n\ \varphi(\alpha,n,\sigma,S)$, and when $f(\alpha,\sigma)\downarrow$, the computation will be finite.

Let $g(\sigma) = W(\lambda\alpha f(\alpha,\sigma))$. If $g(\sigma)\downarrow$ the length of the computation will be at most $\omega$.

Corollary 4. $Th(W,S)$ is not p-normal, i.e. we cannot compare lengths.

Proof. It is not hard to construct a computation in $W$ who's length is greater than $\omega$, and which has a natural number as argument. p-normality and the observation above would yield that the set of computations were computable.

Thus $Th({}^2E)$ and $Th({}^2E,W)$ are different, although they have the same envelopes and the same sections. This contrasts that in the normal case, equality between envelopes gives equivalence between theories.

Observe that 2-en(S,W) will always be closed under $\exists^\omega$. However,

<u>Conjecture.</u> Let $S$ be an arbitrary type-2-functional. In general, the functional

$$\varphi(A,a) \simeq 0 \iff \exists n \in \omega(\langle n,a\rangle \in A)$$

will not be W,S-computable in the sence of Moschovakis i.e. there is no index $e$ such that

$$\{e\}^{W,S}(e',\vec{x}) \simeq 0 \iff \exists n\{e\}^{W,S}(n,\vec{x}) \simeq 0$$

and $$\|\langle e,e',x\rangle\|_{W,S} > \inf\{\|\langle e',n,\vec{x},0\rangle\|_{W,S}\ ;\ n \in \omega\}$$

The conclusion is false when $S = {}^2E$.

P. Aczel proved that the partial functional

$$\varphi(f) \simeq 0 \iff \exists n f(n) \downarrow$$

is not computable in any total functional.

Let $\Pi^1_1(S)\text{-ind} = \{\Gamma^\infty;\ \Gamma$ is a positive $\Pi^1_1(S)$-operator$\}$

$$|\Pi^1_1(S)|_{df} = \mathrm{Sup}\{|\Gamma|;\ \Gamma \text{ is a positive } \Pi^1_1\ S)\text{-operator}\}.$$

Problem. Let $|S,W|$ denote the supremum of the lengths of computations in $Th(S,W)$. Will $|S,W| = |\Pi^1_1(S)|$ ?

Remark. If the conjecture above is disproved for arbitrary $S$, we have a positive solution to the problem, by the first recursion theorem. We will always have $|S,W| \leq |\Pi^1_1(S)|$, since the set of computations is given by a $\Pi^1_1(S)$-inductive definition.

## Bibliography.

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